THE BEHAVIOR OF A CONSERVATIVE SYSTEM UNDER THE ACTION OF SLIGHT FRICTION AND SLIGHT RANDOM NOISE

(O RABOTE KONSERVATIVNOI SISTEMY PRI VOZDEISTVII MALOGO TRENIIA I MALOGO SLUCHAINOGO SHUMA)

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R.Z.KHAS'MINSKII
(Moscow)

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The averaging principle, developed earlier [1 and 2] for Markov processes, is applied for the analysis of a stationary process behavior of a one-dimensional conservative system, under the action of slight random noise of the "white" type and slight nonlinear friction.

In a number of publications the averaging principle was applied to Markov processes. For example, in [3], a linear conservative system, under the action of slight white noise, was investigated. In [4] a similar system, with friction, was studied under the assumption that the effect of noise is much smaller than the effect of friction. An arbitrary one-dimensional conservative system was considered in the interesting monograph [5], under the assumption, however, that the noise intensity is independent of the point of phase space. In addition, in all these works, the rigorous proof of the application of the averaging principle to random processes was not given.

In [6], a linear conservative system subjected to the action of a nonlinear friction and white noise, under various relations between friction and noise, was investigated. The proof of the averaging method for such systems was also given, based on [1 and 2]. In the present paper, the results of [6] are generalized to the case of the one-dimensional conservative system of the general type. In particular, the expression is obtained for the density limit of the stationary probability distribution of the system investigated, when the noise and friction tend to zero. A method is also indicated for the determination of further terms of the asymptotic expansion.

1. The formulation of the problem. We shall consider a mechanical system with one degree of freedom, the undisturbed motion of which is periodic and described by Hamilton's function $H(p,q)$.

Let the system be subjected to the action of slight friction and slight random disturbances of the white noise type. If we assume that the effect of the random disturbances and the work of the friction forces over the period are of the same small orde of magnitude ϵ, then the motion of such a system is decribed by Equations

$$dq = [\partial H / \partial p - \varepsilon f_q (p, q)]\, dt + \sqrt{\varepsilon}\, [\sigma_{11} (p, q)\, d\xi_1 (t) + \sigma_{12} (p, q)\, d\xi_2 (t)]$$

$$dp = [- \partial H / \partial q + \varepsilon f_p (p, q)]\, dt + \sqrt{\varepsilon}\, [\sigma_{21} (p, q)\, d\xi_1 (t) + \sigma_{22} (p, q)\, d\xi_2 (t)] \quad (1.1)$$

In the system (1.1), the functions $\xi_1(t)$ and $\xi_2(t)$ are independent Wiener random processes (integrals of "white noise") such that $\langle \xi (t)\rangle = 0$, $\langle \xi^2 (t)\rangle = t$ (pointed brackets denote probability averaging) the functions f_p

and f_q denote, in general, a nonlinear friction in the system and the matrix $((\sigma_{ij}(p,q))$ is the intensity of the random impulses at the point (p,q) of the phase space. It is not difficult to show that over a time of the order $O(1)$, the solution of the system (1.1) can be as close as desired to the solution of the undisturbed system

$$q' = \partial H / \partial p, \qquad p' = -\partial H / \partial q \tag{1.2}$$

if ε is sufficiently small. In particilar, the energy of the system will not change significantly over this time. However, the relaxation time for the system (1.1) is of the order $1/\varepsilon$. Therefore, we investigate below the behavior of the system (1.1) over this period of time; in particular, we determine the stationary process of the behavior of the system.

2. The equation of energy change of the system. The functions $\xi_i(t)$ do not have a differential in the classical sense, however the system (1.1) can be given an exact meaning, if we understand it as a system of stochastic differential equations of Jto [7 and 8].

As is known, the solution $p_\varepsilon(t)$, $q_\varepsilon(t)$ of the system (1.1), satisfying the initial conditions

$$p_\varepsilon(0) = p_0, \qquad q_\varepsilon(0) = q_0 \tag{2.1}$$

represents a random process in the phase space of the system. Applying Jto's formula and changing variables in the stochastic integral [8], we can write the equation of energy change $E_\varepsilon(t) = H(p_\varepsilon(t), q_\varepsilon(t))$ of the system (1.1). This equation has the form

$$dE_\varepsilon(t) = \varepsilon\left[-f_p(p,q)\frac{\partial H}{\partial q} + f_q\frac{\partial H}{\partial p} + \frac{1}{2}\left(a_{11}\frac{\partial^2 H}{\partial q^2} + 2a_{12}\frac{\partial^2 H}{\partial p\partial q} + a_{22}\frac{\partial^2 H}{\partial p^2}\right)\right]dt +$$
$$+ \sqrt{\varepsilon}\left[\left(\sigma_{11}\frac{\partial H}{\partial q} + \sigma_{21}\frac{\partial H}{\partial p}\right)d\xi_1(t) + \left(\sigma_{12}\frac{\partial H}{\partial q} + \sigma_{22}\frac{\partial H}{\partial p}\right)d\xi_2(t)\right] \tag{2.2}$$
$$(a_{ij} = \sigma_{i1}\sigma_{j1} + \sigma_{i2}\sigma_{j2})$$

We shall investigate Equation (2.2) together with one of the equations of the system (1.1). In the system obtained in such a manner the "fast" and "slow" motions are separated. For such systems (and for much more general ones) the averaging principle is well known. This principle, in the absence of chance (i.e. with $\sigma_{ij} \equiv 0$) was established in the works of Krylov, Bogoliubov and Mitropol'skii. This principle allows, over a period of time of the order of $1/\varepsilon$, to approximate the equation of the slow motion with an equation, the right side of which is averaged over the fast motion (the fast motion in the present case coincides with the motion of the undisturbed system). The corresponding averaging principle, for systems containing chance of the type under consideration here, was developed by the author (see [1 and 2]). Applying it to the present case, we obtain, that the probability distribution for the energy $E_\varepsilon(t)$ of the system (1.1) over a time interval of the order $1/\varepsilon$ with small ε is close to the probability distribution for the one-dimensional Markov random process $E_0(t)$, described by Equation

$$dE_0(t) = \frac{\varepsilon}{T(E_0)}[f^*(E_0)] + F^*(E_0)]\,dt + \sqrt{\varepsilon}\,\frac{\sigma^*(E_0)}{\sqrt{T(E_0)}}\,d\xi(t) \tag{2.3}$$

Here

$$T(E) = \oint\left(\frac{\partial H}{\partial q}\right)^{-1}dq, \qquad \varepsilon\sigma^{*2}(E) = \oint\left\langle\frac{[\Delta H(p_\varepsilon(t), q_\varepsilon(t)]^2}{\Delta t}\right\rangle dt =$$
$$= \varepsilon\oint\left[a_{11}\left(\frac{\partial H}{\partial q}\right)^2 + 2a_{12}\frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + a_{22}\left(\frac{\partial H}{\partial p}\right)^2\right]dt$$

$$f^*(E) = \oint f_p dp + f_q dq, \qquad F^*(E) = \frac{1}{2}\oint\left[a_{11}\frac{\partial^2 H}{\partial q^2} + 2a_{12}\frac{\partial^2 H}{\partial p\partial q} + a_{22}\frac{\partial^2 H}{\partial p^2}\right]dt$$

The integration is carried out over the path of the undisturbed motion $H(p,q) = E$.

From these equations, it can be seen that $T(E)$ is the period of the undisturbed motion, corresponding to the energy E, and $\varepsilon f^*(E)$, $\varepsilon\sigma^{*2}(E)$ and $\varepsilon F^*(E)$ are the work of the friction forces, the local diffusion of the process $H(p_\varepsilon(t), q_\varepsilon(t))$ and the work of the additional friction forces, originating from the diffusion, over the same path $H(p,q) = E$.

Let us denote by $u(E,t)$ the density of the probability distribution of the process $E_0(t)$. It is known, that this density satisfies the equation of Fokker-Plank-Kolmogorov, which in the present case has the form

$$\frac{\partial u}{\partial t} = \varepsilon\left\{\frac{1}{2}\frac{\partial^2}{\partial E^2}\left(\frac{\sigma^{*2}(E)}{T(E)}u\right) - \frac{\partial}{\partial E}\left[(f^*(E) + F^*(E))\frac{u}{T(E)}\right]\right. \tag{2.4}$$

If the initial distribution density is known

$$u(E, 0) = u_0(E) \tag{2.5}$$

then, solving Equation (2.4) with the initial condition (2.5), we can find approximately the probability distribution for the energy of the system (1.1) over the time interval $O(1/\varepsilon)$. The smaller ε, the more exact the approximation. The solution $\mu_0(E)$ of Equation (2.4), independent of time, normalized by condition

$$\int_0^\infty \mu_0(E)\,dE = 1 \tag{2.6}$$

gives the probability density of the stationary process (2.3), if it exists.

The function $\mu_0(E)$ is easily calculated. For this purpose we note, and this is easy to prove, that the functions σ^{*2} and F^* are connected by the relation

$$\frac{\partial}{\partial E}\sigma^{*2} = 2(F^* + \Phi) \qquad \left(\Phi(E) = \frac{1}{2}\oint_{H=E}\frac{\partial a_{22}}{\partial p}dq - \frac{\partial a_{11}}{\partial q}dp\right) \tag{2.7}$$

Using (2.7), we obtain

$$\mu_0(E) = cT(E)\exp\left\{2\int_0^E \frac{f^*(z) - \Phi(z)}{\sigma^{*2}(z)}dz\right\} \tag{2.8}$$

where the normalizing constant c is found from (2.6), and it is assumed that σ^{*2} does not become zero, so that the process (2.3) is ergodic.

In the particular case $\Phi(E) = 0$

$$\mu_0(E) = cT(E)\exp\left\{2\int_0^E \frac{f^*(z)}{\sigma^{*2}(z)}dz\right\} \tag{2.9}$$

The condition $\Phi(E) = 0$ seems somewhat artificial, however it is satisfied in the important particular case, when $H = {}^1/_2\,p^2/m + U(q)$, and the random noise depends only on the position of the q-particle and effects directly only the impulse of the system. The system (1.1) then becomes

$$dq = \frac{p}{m}dt, \qquad dp = \left[-\frac{dU}{dq} + \varepsilon f(p, q)\right]dt + \sqrt{\varepsilon}\,\sigma(q)\,d\xi(t) \tag{2.10}$$

Since $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$, and $\sigma_{22} = \sigma(q)$ in this case, then $a_{11} = 0$, $a_{22} = \sigma^2(q)$. Therefore the condition $\Phi(E) = 0$ is satisfied.

3. The stationary state of the behavior of the system. Since the coefficients of the system (1.1) are independent of time, it is natural to expect that, when $t \to \infty$, a definite limiting state is established, if the general assumptions of the friction in the system are fulfilled. It is of interest to study this limiting state as $\varepsilon \to 0$.

Let us state the problem more precisely. Let $\mu^{(\varepsilon)}(p, q)$ be the density on the measure $dp\,dq$ of the stationary probability distribution for the Markov process (1.1), i.e. such a function, that

$$\iint \mu^{(\varepsilon)}(p_1, q_1)\,P_\varepsilon(p, q, t, p_1, q_1)\,dp_1\,dq_1 = \mu^{(\varepsilon)}(p, q), \qquad \iint \mu^{(\varepsilon)}(p, q)\,dp\,dq = 1 \tag{3.1}$$

Here $P_\varepsilon(p, q, t, p_1, q_1)$ is the probability density of the transition from point (p,q) to point (p_1,q_1) in time t for the process (1.1). We shall find the limit of the function $\mu^{(\varepsilon)}(p, q)$ as $\varepsilon \to 0$.

It is easy to show from Liouville's theorem, that the limit $\mu^{(0)}(p, q)$, if it exists, actually depends only on one variable $\mu^{(0)}(p, q) = \mu^{(0)}(H(p, q))$.

Let the following conditions be satisfied:

a) with any $\varepsilon > 0$ there exists a stationary distribution for the process (1.1), the density of which $\mu^{(\varepsilon)}$ on the measure $dp\,dq$ satisfies relation (3.1);

b) the distribution $\mu^{(\varepsilon)}$ "does not spread out" as $\varepsilon \to 0$, i.e. for any δ there is a c such that

$$\iint\limits_{H(p,q)>c} \mu^{(\varepsilon)}(p, q)\,dp\,dq < \delta \tag{3.2}$$

for all $\varepsilon > 0$. Then, using arguments analogous to [6], it can be shown, that a low limit of the function $\mu^{(\varepsilon)}(p, q)$ as $\varepsilon \to 0$ differs from the stationary solution of the problem (2.4), (2.6) only by the multiplier $1/T(H)$, i.e.

$$\mu^{(0)}(p, q) = \mu_0(H(p, q)) / T(H(p, q))$$

The multiplier $1/T(H)$ arises from the fact that $\mu^{(0)}$ is the density on the measure $dp\,dq$ and μ_0 is the density on the measure dE.

Using (2.8), we obtain

$$\mu^{(0)}(p, q) = c \exp\left\{2 \int\limits_0^{H(p,q)} \frac{f^*(z) - \Phi(z)}{\sigma^{*2}(z)}\,dz\right\} \tag{3.3}$$

This result enables us to calculate approximate values of the important parameters of the process (1.1) for small values of ε. For example, the average energy of vibration $\langle E_\varepsilon \rangle$ as $\varepsilon \to 0$ tends to the limit

$$\langle E_0 \rangle = \int E \mu_0(E)\,dE$$

It is interesting to note, that when condition $\Phi(E) = 0$ is satisfied, the extremes of the function $\mu^{(0)}(p, q)$ are reached on those trajectories of the undisturbed motion $H(p,q) = E_0$ for which E_0 coincides with the roots of Equation

$$f^*(E_0) = 0 \tag{3.4}$$

Equation (3.4) is the well known condition for the determination of the spacing of the limit cycle in a system with slight friction, consisting in the requirement, that the work of the friction forces during the limit cycle should be equal to zero. Thus, the stable and unstable limit cycles of a system without chance correspond to the maxima and minima of the density of the invariant measure for a system with white noise, if the condition $\Phi(E) = 0$ is satisfied.

Probably, the most practical case is the one, when the random noise is much smaller than the friction, i.e. $a_{ij} \ll 1$. Let $a_{ij}(p, q) = \mu a_{ij}^\circ(p, q)$, where $\mu \ll 1$ is a parameter, describing the ratio of the random noise work to the work of the friction forces over the period of the motion. Then also $\Phi(E) = \mu\Phi^\circ(E)$, $\sigma^{*2}(E) = \mu\sigma^{\circ 2}(E)$, where the functions $\Phi^\circ(E)$ and $\sigma^{\circ 2}(E)$ are determined on a_{ij}° in the same way as Φ and σ^{*2} are determined on a_{ij}. It can be seen from (3.3), that when $\mu \ll 1$ the stationary distribution of the process tends to the highest maximum of the function

$$U(E) = \int\limits_0^E \frac{f^*(z)}{\sigma^{\circ 2}(z)}\,dz$$

i.e. to one of the limit cycles of the system without chance. If the function has a number of maxima of equal height, the case has also been analyzed in detail (see [9]).

In conclusion, we shall make some comments:

1) Conditions (a) and (b) in Section 3, are very important in the proof of the above mentioned results. We can give sufficient conditions for their satisfaction in terms of Liapunov's functions, based on the results [10].

2) The method described here can also be applied to the analysis of multi-dimensional systems. However for such systems there exists, as a rule, more than one integral of motion. Therefore the "slow motion" will be here, in general, multi-dimensional, and the method presented will lead only to a reduction in the dimensions of the problem. For example, the density limit of the stationary distribution will depend not only on the energy, but also on all the other integrals of motion, and therefore explicit equations can only be obtained as an exception.

3) The function $\mu^{(0)}$ is only the first term of the asymptotic expansion

$$\mu^{(\varepsilon)}(p, q) = \mu^{(0)} + \varepsilon\mu^{(1)} + \cdots$$

Using another approach, analogous to the one suggested in [11], we can also obtain the other terms of this asymptotic expansion. In addition, it is understood, the functions $\mu^{(1)}$ with $n \geqslant 1$ will depend not only on the energy.

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